# Pochhammer Symbols, q-Analogs, Gaussian Binomial Coefficients & Bell Polynomials

Nicholas Wheeler
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**Introduction.** In recent work[1] I encountered a connection in which the partition problem (enumeration of the partitions of an integer) and the theory of permutations collaborate in a way that was unfamiliar to me. Web doodling led me somehow to the Wikipedia article "Gaussian binomial coefficient," which summarizes properties of objects I had never previously encountered (so rare that they are not indexed in Abramowitz & Stegun), which are described as "q-analogs of the binomial coefficients" (the "q-analog" concept is also one of which I have managed to live thus far in total ignorance) and said to "occur in the counting of symmetric polynomials and in the theory of partitions"—topics that plausibly related fairly directly to the Lipsky work that inspired me. My objective here will be to familiarize myself with the basics of those several novel notions.

**Pochhammer symbols/polynomials.** Pochhammer symbols are a notational device—a generalization of $n!$—commonly attributed to Leo Pochhammer (1841–1920) because of the use he made of them in his work on special functions (particularly hypergeometric functions), but the associated Pochhammer polynomials had been studied already (1730) by James Stirling (1692–1770). Terminology and notation in this subject area remains wonderfully diverse (see the Wikipedia article mentioned above). I adopt Pochhammer's own notation

$$\left.\begin{aligned} &(x)_n \equiv \underbrace{x(x+1)(x+2)\cdots(x+n-1)}_{n\text{ factors}} \quad : \quad n \geqslant 1 \text{ an integer} \\ &(x)_0 \equiv 1 \end{aligned}\right\} \tag{1}$$

[1] "Working notes concerning a paper by Lester Lipsky" (November 2016), pages 42–44.

because that conforms to the convention adopted by Abramowitz & Stegun,[2] by the authors of *Mathematica*, and by Spanier & Oldham,[3] whose Chapter 18 provides an elaborately detailed survey of this subject. Graham, Knuth & Patashnik[4] refer to $(x)_n$ as "$x$ to the $n$ rising" (which they denote $x^{\bar{n}}$). Others refer simply to the "rising factorial."[5]

A generating function for the $(x)_n$ is (as *Mathematica* confirms)

$$(1-t)^{-x} = \sum_{n=0}^{\infty} \tfrac{1}{n!}(x)_n t^n \tag{2}$$

while the falling factorials are generated by $(1+t)^x$.

From the definition of $(x)_n$ it follows directly that $(x)_k = (x)_n(x+n)_{k-n}$; *i.e.*, that

$$(x)_{m+n} = (x)_n(x+n)_m = (x)_m(x+m)_n \tag{3}$$

and that

$$(1)_n = n! \tag{4}$$

while it follows as a corollary from (3) that

$$\frac{(x+m)_n}{(x)_n} = \frac{(x+n)_m}{(x)_m} \tag{5}$$

In the case $m=1$ this gives the recursion relation with respect to argument

$$(x+1)_n = \Big(\frac{x+n}{x}\Big)\cdot(x)_n \tag{6.1}$$

Also

$$\begin{aligned}(x)_{n+1} = (x)_n(x+n)_1 &= (x+n)(x)_n \\ &= x\,(x+1)_n\end{aligned} \tag{6.2}$$

which provides recursion with respect to degree.[6]

Theory of the gamma function (Spanier & Oldham, 43:5:10, page 436) supplies

$$\frac{\Gamma(x+n)}{\Gamma(x)} = (x)_n \quad : \quad n = 1, 2, 3, \ldots \tag{7.1}$$

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[2] *Handbook of Mathematical Functions* (1964).

[3] *An Atlas of Functions* (1987).

[4] *Concrete Mathematics* (1994).

[5] Write the factors that define the "falling factorial" (which Graham *et al.* call "$x$ to the $n$ falling" and denote $x^{\underline{n}}$) in reverse order and it becomes a rising factorial:

$$\underbrace{x(x-1)(x-2)\cdots(x-n+1)}_{n\text{ factors}} = (x-n+1)_n$$

The expression on the left can be written $(-)^n(-x)_n$ so one has the "reflection formula"

$$(-x)_n = (-)^n(x-n+1)_n$$

[6] In point of terminology: $(x)_n = (\text{argument})_{\text{degree}}$.

which written

$$(x)_y = \frac{\Gamma(x+y)}{\Gamma(x)} \tag{7.2}$$

serves to relax the stipulation that $n$ be a non-negative integer.[7] This extended definition permits one on the basis of (3) to write $(x)_0 = (x)_n(x+n)_{-n}$ or

$$\frac{1}{(x)_n} = (x+n)_{-n} \tag{8}$$

Familiarly

$$\Gamma(n) = (n-1)! \quad : \quad n = 1, 2, 3, \ldots$$

so (7.2) can be written

$$(x)_y = \frac{(x+y-1)!}{(x-1)!} = y!\binom{x+y-1}{y} \quad : \quad \{x, y\} = 1, 2, 3, \ldots \tag{9}$$

As the factorial and binomial coefficient are implemented by *Mathematica* this equation remains valid even when $\{x, y\}$ are non-integral (any real numbers). "Vandermonde's theorem" asserts that

$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k} \tag{10}$$

which can reportedly[8] be obtained as a corollary of the Chu-Vandermonde identity

$$\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k}$$

which is a non-integral extension of the Vandermonde's combinatorial identity ("Vandermonde's convolution," 1772), which appears already in *Jade Mirror of the Four Elements* (1303), by Chu Shih-Chieh.[9] Equation (10) resembles the binomial theorem, except that the familiar exponents have become subscripts; this kind of replacement is the hallmark of the "umbral calculus," which feeds on the observation that formal parallels similar to (10) arise also from (for example) the theory of Hermite polynomials (S & O 25:5:2, page 219) and Bernoulli polynomials (S & O 19:5:3, page 169). But in other ways the Pochhammer polynomials—which arise when $n$ is a non-negative integer, and of which the

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[7] This evidently is the extended definition built into *Mathmatica*, which responds to the command `Pochhammer[x,n]` whether or not $n$ is a non-negative integer.

[8] See the Wikipedia articles "Vandermonde's identity."

[9] For a fascinating account of the amazing accomplishments of Chu and his predecessors, see the Wikipedia article "Chinese mathematics."

first few are

$$
\begin{aligned}
(x)_0 &= 1\\
(x)_1 &= x\\
(x)_2 &= x^2 + x\\
(x)_3 &= x^3 + 3x^2 + 2x\\
(x)_4 &= x^4 + 6x^3 + 11x^2 + 6x\\
(x)_5 &= x^5 + 10x^4 + 35x^3 + 50x^2 + 24x
\end{aligned}
$$

—stand oddly apart: they do not arise from a differential equation, do not possess orthogonality properties, do not possess natural relationships to other families of named polynomials. One does, however, have[10] the identity

$$(x)_m(x)_n = \sum_{k=0}^{\text{lesser}\{m,n\}} \binom{m}{k}\binom{n}{k} k!(x)_{m+n-k}$$

which places one in position to write

$$\big(\textstyle\sum_m a_m(x)_m\big)\big(\textstyle\sum_n b_n(x)_n\big) = \textstyle\sum_k c_{mn,k}(x)_k$$

which is to say: the falling factorials are elements of a polynomial ring. The "connection coefficients" are confirmed by *Mathematica* to be describable

$$
\begin{aligned}
\binom{m}{k}\binom{n}{k}k! &= \frac{(m)_k(n)_k}{k!}\\
&= \begin{cases}\text{number of distinct ways one can select } k\ \alpha\text{s and } k\ \beta\text{s}\\ \text{from } \{\alpha_1,\alpha_2,\ldots,\alpha_m\}, \{\beta_1,\beta_2,\ldots,\beta_n\}.\end{cases}
\end{aligned}
$$

Google reports the existence of literature relating to various "generalized Pochhammer symbols/polynomials," some of which may be of future interest because they involve the partitions $p(n)$ of integers $n$.

**The "q-analog" concept.** The idea here goes back to Euler, and was cultivated also by Gauss. One has

$$\lim_{q\to 1}\frac{1-q^n}{1-q} = n \quad : \quad \text{all } n \text{, whether integral, real or complex}$$

which suggests that, in formulæ—especially those encountered in combinatorics and special function theory—where integers occur, one might profitably contemplate replacements

$$n \longrightarrow [n]_q \equiv \frac{1-q^n}{1-q} = \underbrace{1 + q + q^2 + \cdots + q^{n-1}}_{n \text{ terms}} \tag{11}$$

—at least in cases where that adjustment does not do too much damage to preexisting formal relationships.

[10] Here $(x)_n$ denotes the *falling* factorial.

It becomes, in this light, natural to consider the “$q$-factorial”

$$\begin{aligned}[n]_q! &= [1]_q \cdot [2]_q \cdots [n-1]_q \cdot [n]_q \\ &= \frac{1-q}{1-q} \cdot \frac{1-q^2}{1-q} \cdots \frac{1-q^{n-1}}{1-q} \cdot \frac{1-q^n}{1-q} \\ &= 1 \cdot (1+q) \cdots (1+q+\cdots+q^{n-2}) \cdot (1+q+\cdots+q^{n-1})\end{aligned} \tag{12}$$

As remarked in the Wikipedia article “q-analog,” while the factorial $n!$ counts the number of permutations of $\{1, 2, \ldots, n\}$, the q-factorial enumerates the inversions in the list of permutations. For example, command

```
Perms4=Permuatations[{1,2,3,4}]
```

to create a list of the 24 permutations of $\{1, 2, 3, 4\}$. The command

```
Tally[Table[Inversions[Perms4[[k]]],{k,1,24}]
```

produces[11]

$$\{\{0,1\},\{1,3\},\{2,5\},\{3,6\},\{4,5\},\{5,3\},\{6,1\}\}$$

from which we learn, for example, that 5 of the permutations require 2 inversions to be restored to canonical order. Observe now that

```
qFactorial[4,q]//Simplify//Expand
```

produces

$$(1+q)(1+q+q^2)(1+q+q^2+q^3) = 1+3q+5q^2+6q^3+5q^4+3q^5+q^6$$

in which the exponents and coefficients conform precisely to the pattern in the tallyed list of inversions. And since every permutation contributes once to that list, we are not surprised to find (set $q = 1$) that $1+3+5+6+5+3+1 = 24$. For arbitrary $n$ we expect on this basis to have

$$\sum_{\wp} q^{\text{inversions}(\wp)} = [n]_q! \tag{13}$$

where $\wp$ ranges over the permutations of $\{1, 2, \ldots, n\}$. But of this result—demonstrated above in the case $n = 4$ and asserted in the literature—I am not in position to provide a general proof.

Looking to the $q$-analog of the monomial $x^n$ we might expect to write

$$x^n \longrightarrow [x^n]_q = x^{[n]_q}$$

giving $$\tfrac{d}{dx}x^n = nx^{n-1} \longrightarrow \tfrac{d}{dx}[x^n]_q = [n]_q x^{[n]_q - 1}$$

The formal substance of $q$-analog mathematics is, however, better served if one introduces a “$q$-derivative,” writing

$$\big(\tfrac{d}{dx}\big)_q x^n = [n]_q x^{n-1} \tag{14}$$

[11] The symmetry evident here is a general feature of such lists.

If the "$q$-exponential" is defined

$$e_q(x) \equiv \sum_{n=0}^{\infty} \frac{1}{[n]_q!} x^n \tag{15.1}$$

one then has

$$\big(\tfrac{d}{dx}\big)_q e_q(x) = e_q(x) \tag{15.2}$$

From definitions

$$\begin{aligned} \sin_q(x) &\equiv x - \frac{1}{[3]_q!}x^3 + \frac{1}{[5]_q!}x^5 - \frac{1}{[7]_q!}x^7 + \cdots \\ \cos_q(x) &\equiv 1 - \frac{1}{[2]_q!}x^2 + \frac{1}{[4]_q!}x^4 - \frac{1}{[6]_q!}x^6 + \cdots \end{aligned}$$

one is led similarly to

$$\begin{aligned} \big(\tfrac{d}{dx}\big)_q \sin_q(x) &= +\cos_q(x) \\ \big(\tfrac{d}{dx}\big)_q \cos_q(x) &= -\sin_q(x) \end{aligned}$$

which illustrate the "persistence of formal relationships" which $q$-analog mathematics strives to maintain. There are, however, surprises: with the assistance of *Mathematica* we satisfy ourselves that[12]

$$\sum_{k=0}^{n} (-)^k \frac{1}{[k]_q![n-k]_{1/q}!} = \begin{cases} 1 & : \quad n = 0 \\ 0 & : \quad n = 1, 2, 3, \ldots \end{cases}$$

from which it follows that

$$[e_q(x)]^{-1} \text{ is given not by } e_q(-x) \text{ but by } e_{1/q}(-x) \tag{15.3}$$

As the preceding examples illustrate, the construction of useful/informative $q$-analogs is not generally a matter of straightforward application of (11)—if it were one would expect the $q$-analog of the Pochhammer symbol (1) to be defined by the construction

$$x(x + [1]_q)(x + [2]_q) \cdots (x + [n-1]_q)$$

—but it isn't: by universal convention the $q$-Pochhammer symbol is denoted/defined

$$\left.\begin{aligned} (a;q)_n &= \prod_{k=0}^{n-1}(1 - aq^k) = \underbrace{(1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})}_{n \text{ factors}} \\ (a;q)_0 &= 1 \end{aligned}\right\} \tag{16}$$

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[12] Again I possess no explicit proof, but note that

$$[n]_{1/q}! = \frac{[n]_q!}{q^{\Delta(n)}}$$

where $\Delta(n)$ denotes the triangular number $\sum_{i=1}^{n-1} i = \frac{1}{2}n(n-1)$.

From (16) we obtain

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

and Euler’s function[13]

$$\phi(q) = (q;q)_\infty = \prod_{k=1}^{\infty}(1-q^k) = (1-q)(1-q^2)(1-q^3)\cdots$$

From

$$\begin{aligned}\frac{(q\,;q)_n}{(1-q)^n} &= \frac{1-q}{1-q}\frac{1-q^2}{1-q}\frac{1-q^3}{1-q}\cdots\frac{1-q^n}{1-q}\\ &= [1]_q[2]_q[3]_q\cdots[n]_q\\ &= [n]_q! \end{aligned} \tag{17}$$

we obtain the anticipated relationship between Pochhammer’s “rising factorial” and the $q$-Pochhammer symbols.

It was remarked on page 5 that $[n]_q$ some fairly recondite things to say about the permutations of $\{1, 2, \ldots, n\}$. Similarly recondite information about the partitions of $n$ can be obtained from $(a\,;q)_\infty$.[14] It is asserted, for example, that

$$\frac{1}{(a\,;q)_\infty} = \sum_{m,n=0}^{\infty} p_{m,n}a^m q^n$$

$$p_{m,n} = \begin{cases}\text{number of ways } n \text{ can be partitioned}\\ \text{into } m \text{ or fewer parts}\end{cases}$$

which I must again be content to demonstrate by example. The command `IntegerPartitions[6,3]` lists the partitions of 6 into 3 or fewer parts:

$$\{\{6\},\{5,1\},\{4,2\},\{4,1,1\},\{3,3\},\{3,2,1\},\{2,2,2\}\}$$

of which there are $p_{3,6} = $ `Length[IntegerPartitions[6,3]]` $= 7$. Proceeding in this way, we construct the following table:

$$\begin{array}{ll} p_{1,6} = 1 & p_{5,6} = 10\\ p_{2,6} = 4 & p_{6,6} = 11\\ p_{3,6} = 7 & p_{7,6} = 11\\ p_{4,6} = 9 & p_{8,6} = 11 \end{array}$$

[13] The *Mathematica* commands

`QPochhammer[a,q,n]` produce $(a;q)_n$
`QPochhammer[a,q]` produce $(a;q)_\infty$
`QPochhammer[q]` produce $(q\,;q)_\infty$

[14] See the Wikipedia article “q-Pochhammer symbol,” which provides a rich store of identities.

where the repeated 11s occur because 6 cannot be partitioned into more than 6 parts: $6 = 1+1+1+1+1+1$; the answer to the question $p_{m,n}$ $(m > n)$ is implicit in the answer to the question $p_{6,6}$. Looking now to the second half of the assertion, the command

```
SeriesCoefficient[Series[f(x),{x,0,m],n]   :   m > n
```

produces the coefficient of $x^n$ in the expansion of $f(x)$, and supplies the information that (in particular) the coefficient of $q^6$ in the $q$-expansion of $(a;q)_\infty^{-1}$ is

$$\begin{aligned}&\frac{a+3a^2+3a^3+2a^4+a^5+a^6}{1-a}\\ &\qquad\qquad = a+4a^2+7a^3+9a^4+10a^5+11a^6+11a^7+11a^8+\cdots\end{aligned}$$

The coefficients are seen to be precisely those that appear in the preceding table, and it is now evident that the repeated 11s can be attributed to the repetitive structure of $(1-a) = 1+a+a^2+a^3+\cdots$.

$q$-Pochhammer symbols enter frequently into the construction of $q$-analogs. For example, one has[14]

$$\varGamma_q(x) = \frac{(1-q)^{1-x}(q;q)_\infty}{(q^x;q)_\infty}$$

—a definition justified by the fact that it entails

$$\varGamma_q(x+1) = [x]_q\varGamma_q(x) \quad : \quad \text{any } x \tag{18.1}$$

$$\varGamma_q(n+1) = [n]_q! \quad : \quad n \text{ any non-negative integer} \tag{18.2}$$

**Gaussian binomial coefficients.** These are straightforward $q$-analogs of ordinary binomial coefficients, denoted/defined by

$$\begin{aligned}\binom{n}{m}_q &= \frac{[n]_q!}{[m]_q![n-m]_q!} &(19.1)\\ &= \begin{cases}\dfrac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q)(1-q^2)\cdots(1-q^m)} & : \quad m \leqslant n\\ 0 & : \quad m > n\end{cases}\end{aligned}$$

which by (17) and (18.1) can also be described

$$= \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}} \tag{19.2}$$

$$= \frac{\varGamma_q(n+1)}{\varGamma_q(m+1)\varGamma_q(n-m+1)} \tag{19.3}$$

and give back ordinary binomial coefficients in the limit $q \to 1$. Ordinary binomial coefficients—whether considered to arise from $(x+y)^n$ or in response to questions like "in how many ways can $m$ objects be selected from $n$, irrespective of order?" or "in how many ways can $m$ 1s and $n-m$ 0s be arranged?"—are obviously integers, though

$$\binom{n}{m} = \frac{n(n-1)\cdots(n-m+1)}{1\cdot 2\cdot 3\cdot \cdots \cdot m} \quad : \quad n \text{ and } m \leqslant n \text{ integers}$$

gives every appearance of being a fraction.[15] Similarly, (19.1) seems to indicate that Gaussian binomial coefficients are ratios of polynomials, but in fact they are (again, the demonstration eludes me) polynomials, as the literature claims and *Mathematica* confirms:

$$\begin{aligned}&\texttt{Series[QBinomial[5,2,q],\{q,0,8\}]//Normal}\\ &\quad = \texttt{Series[}\frac{\texttt{(1-q}^5\texttt{)(1-q}^4\texttt{)}}{\texttt{(1-q)(1-q}^2\texttt{)}}\texttt{,\{q,0,8\}]//Normal}\\ &\quad = 1 + q + 2q^2 + 2q^3 + 2q^2 + q^5 + q^6\end{aligned}$$

Thus

$$\begin{aligned}\texttt{QBinomial[6,0,q]} &= 1\\ \texttt{QBinomial[6,1,q]} &= 1 + q + q^2 + q^3 + q^4 + q^5\\ \texttt{QBinomial[6,2,q]} &= 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8\\ \texttt{QBinomial[6,3,q]} &= 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^3 + 3q^6 + 2q^7 + q^8 + q^9\\ \texttt{QBinomial[6,4,q]} &= 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8\\ \texttt{QBinomial[6,5,q]} &= 1 + q + q^2 + q^3 + q^4 + q^5\\ \texttt{QBinomial[6,6,q]} &= 1\end{aligned}$$

which are seen to conform to the familiar symmetry evident in (19.1)

$$\binom{n}{m}_q = \binom{n}{n-m}_q \tag{20}$$

and also in each instance to possess symmetrically deployed integral coefficients. Extrapolation from a population of illustrative cases leads to the inference that

$$\text{degree } \delta(n,m) \text{ of } \binom{n}{m}_q = m(n-m) \quad : \quad m = 0, 1, 2, \ldots, n \tag{21}$$

which in the case $n = 6$ produces (as above) the sequence $\{0, 5, 8, 9, 8, 5, 0\}$.

*Mathematica*'s implementation of the Gaussian binomial coefficients appears to be based upon some variant of (19.3), for `QBinomial[x,y,q]` gives plottable numerical results even when $\{x, y\}$ are not integers. *Mathematica* struggles when asked to produce more than the first few terms in the $q$-expansion of `QBinomial[x,y,q]`, and appears to be attempting to construct not a polynomial but an infinite series with complex coefficients.

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[15] Curiously, I have been unable to discover or divise a *demonstration* that such ratios assume only integral values.

Here I tabulate the values of `QBinomial[n,m,`$\frac{1}{2}$`]` for $n = 0, 1, 2, \ldots, 6$ and $m = 0, 1, \ldots, n$:

$$\begin{array}{ccccccc}
1 \\
1 & 1 \\
1 & \frac{3}{2} & 1 \\
1 & \frac{7}{4} & \frac{7}{4} & 1 \\
1 & \frac{15}{8} & \frac{35}{16} & \frac{15}{8} & 1 \\
1 & \frac{31}{16} & \frac{155}{64} & \frac{155}{64} & \frac{31}{16} & 1 \\
1 & \frac{63}{32} & \frac{651}{256} & \frac{1395}{512} & \frac{651}{256} & \frac{63}{32} & 1
\end{array}$$

At $q = 1$ we recover the familiar Pascal triangle, while at $q = \frac{3}{2}$ we have

$$\begin{array}{ccccccc}
1 \\
1 & 1 \\
1 & \frac{5}{2} & 1 \\
1 & \frac{19}{4} & \frac{19}{4} & 1 \\
1 & \frac{65}{8} & \frac{247}{16} & \frac{65}{8} & 1 \\
1 & \frac{211}{16} & \frac{2743}{64} & \frac{2743}{64} & \frac{211}{16} & 1 \\
1 & \frac{665}{32} & \frac{28063}{256} & \frac{96005}{512} & \frac{28063}{256} & \frac{665}{32} & 1
\end{array}$$

Note the persistence of the denominators (in these cases all powers of 2) and the familiar bilateral symmetries. It is reported in the literature[16] (and readily confirmed by computation) that

$$\binom{n}{m}_q = \begin{cases} q^m \dbinom{n-1}{m}_q + \phantom{q^{n-m}} \dbinom{n-1}{m-1}_q \\ \phantom{q^m} \dbinom{n-1}{m}_q + q^{n-m} \dbinom{n-1}{m-1}_q \end{cases} \tag{22}$$

which gave back the Pascal identity in the limit $q \to 1$. One can expect most of the numerous binomial identities to have fairly direct Gaussian analogs.

Gaussian binomial coefficients are sometimes called "Gaussian polynomials," and their properties/interrelationships looked upon as relationships among polynomials.[17] For example (22)—which in the case $n = 6$, $m = 2$ reads

$$\binom{6}{2}_q = \begin{cases} q^2 \dbinom{5}{2}_q + \phantom{q^4} \dbinom{5}{1}_q \\ \phantom{q^2} \dbinom{5}{2}_q + q^4 \dbinom{5}{1}_q \end{cases}$$

[16] See, for example, the Wikipedia article "Gaussian binomial coefficient."

[17] I will, as a typographic convenience, allow myself to write $G_{n,m}(q)$ when it is my intent to emphasize the polynomial character of $\binom{n}{m}_q$.

—becomes

$$\begin{aligned}
1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8 \\
&= q^2(1 + q^2 + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6) \\
&\qquad + (1 + q + q^2 + q^3 + q^4) \\
&= (1 + q^2 + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6) \\
&\qquad + q^4(1 + q + q^2 + q^3 + q^4)
\end{aligned}$$

One could, by this recursive process, construct a "Pascal triangle" of Gauss polynomials/coefficients.

I describe now a combinatorial argument[16] that provides an interpretation of the Gauss polynomials $G_{n,m}(q)$ and leads to their direct construction. Let $\mathcal{G}_{n,m}$ be a rectangular grid: height $m$, base $n - m$, area $m(n - m)$.[18] Shortest paths from the lower left to upper right corners of $\mathcal{G}_{n,m}$ involve necessarily $(n - m)$ steps $\rightarrow$ to the right (called "1-steps") and $m$ steps $\uparrow$ up (called "0-steps). One such path associates with each of the permutations $\wp$ of

$$\{\underbrace{1, 1, \ldots, 1}_{n-m}, \underbrace{0, 0, \ldots, 0}_{m}\}$$

of which there are $\binom{n}{m}$. I employ the symbol $\wp$ to denote either a path or the permutation that generates it, and write $\alpha(\wp)$ to denote the area under $\wp$. I describe now, by way of illustration, the constructions that enter into the assembly of $G_{6,2}(q)$; here $\bullet$ signifies unit area under $\wp$:

$$\begin{array}{llll}
\wp_1 & : \quad \{1,1,1,1,0,0\} & \begin{pmatrix} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \end{pmatrix} & \alpha = 0 \\
\wp_2 & : \quad \{1,1,1,0,1,0\} & \begin{pmatrix} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \bullet \end{pmatrix} & \alpha = 1 \\
\wp_3 & : \quad \{1,1,1,0,0,1\} & \begin{pmatrix} \circ & \circ & \circ & \bullet \\ \circ & \circ & \circ & \bullet \end{pmatrix} & \alpha = 2 \\
\wp_4 & : \quad \{1,1,0,1,1,0\} & \begin{pmatrix} \circ & \circ & \circ & \circ \\ \circ & \circ & \bullet & \bullet \end{pmatrix} & \alpha = 2 \\
\wp_5 & : \quad \{1,1,0,1,0,1\} & \begin{pmatrix} \circ & \circ & \circ & \bullet \\ \circ & \circ & \bullet & \bullet \end{pmatrix} & \alpha = 3 \\
\wp_6 & : \quad \{1,0,1,1,1,0\} & \begin{pmatrix} \circ & \circ & \circ & \circ \\ \circ & \bullet & \bullet & \bullet \end{pmatrix} & \alpha = 3 \\
\wp_7 & : \quad \{1,1,0,0,1,1\} & \begin{pmatrix} \circ & \circ & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \end{pmatrix} & \alpha = 4 \\
\wp_8 & : \quad \{1,0,1,1,0,1\} & \begin{pmatrix} \circ & \circ & \circ & \bullet \\ \circ & \bullet & \bullet & \bullet \end{pmatrix} & \alpha = 4 \\
\wp_9 & : \quad \{0,1,1,1,1,0\} & \begin{pmatrix} \circ & \circ & \circ & \circ \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix} & \alpha = 4
\end{array}$$

[18] See again (21), where $m(n - m)$ was encountered in what then seemed to be quite another connection.

$$\begin{array}{llcl}
\wp_{10} & : \quad \{1,0,1,0,1,1\} & \begin{pmatrix} \circ & \circ & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \end{pmatrix} & \alpha = 5 \\[6pt]
\wp_{11} & : \quad \{0,1,1,1,0,1\} & \begin{pmatrix} \circ & \circ & \circ & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix} & \alpha = 5 \\[6pt]
\wp_{12} & : \quad \{1,0,0,1,1,1\} & \begin{pmatrix} \circ & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \end{pmatrix} & \alpha = 6 \\[6pt]
\wp_{13} & : \quad \{0,1,1,0,1,1\} & \begin{pmatrix} \circ & \circ & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix} & \alpha = 6 \\[6pt]
\wp_{14} & : \quad \{0,1,0,1,1,1\} & \begin{pmatrix} \circ & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix} & \alpha = 7 \\[6pt]
\wp_{15} & : \quad \{0,0,1,1,1,1\} & \begin{pmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{pmatrix} & \alpha = 8
\end{array}$$

The claim is that in general

$$G_{n,m}(q) = \sum_{\wp} q^{\,\alpha(\wp)} \tag{23.1}$$

$$= \sum_{\alpha} \#(\alpha) q^{\,\alpha} \tag{23.2}$$

where $\#(\alpha)$ denotes the number of paths that subtend area $\alpha$. In the instance at hand this gives

$$G_{6,2}(q) = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8$$

in precise agreement with the development of `QBinomial[6,2,q]` reported on page 9. The argument that led to (23)[19] makes quite clear the origin of (21); *i.e.*, why it is that $G_{n,m}(q)$ is a polynomial of degree $\delta = m(n-m)$ into which $q^{\,\delta}$ enters with unit coefficient, and also how it comes about that

$$G_{m,n}(1) = \text{sum of coefficients} = \binom{n}{m}$$

I have been unable to discover any reference to the context within which Gauss was led to the invention of Gaussian binomial coefficients/polynomials (suppose it had to do with his hypergeometric work), or any indication of the grounds on which the $\wp$-method is naturally motivated.

One expects the straightforward adjustment that led (page 8) from ordinary binomial coefficients to their Gaussian $q$-analogs to lead similarly to $q$-analogs of the multinomial coefficients.

---

[19] Which I am tempted to call the "path integral method," and which brings to mind the many contexts that involve constructions of the form $\oint_{\text{loop}}$, the "loops" in the present contest being joins of $\{1,1,\ldots,1,0,0,\ldots 0\}$ and its permutations $\wp$.

**Bell polynomials.** I approach this subject as I first encountered it, which was—quite unexpectedly—while preparing some undergraduate lectures on probability theory.[20] Proceeding in the rough informality appropriate to that occasion. . .

Let $p(x)$ be a probability distribution function defined in the entire real line, and assume it to be the case that all moments

$$m_k = \int x^k p(x)dx = \langle x^k \rangle$$

exist.[21] The "moment generating function" is

$$f(t) = \sum_{k=0}^{\infty} \tfrac{1}{k!} m_k t^k = \int e^{xt} p(x)dt = \langle e^{xt} \rangle$$

but often more convenient/informative is the "characteristic function"

$$\begin{aligned}\varphi(t) = \sum_{k=0}^{\infty} \tfrac{1}{k!} m_k (it)^k &= \int e^{ixt} p(x)dt = \langle e^{ixt} \rangle \\ &= \text{Fourier transform of } p(x)\end{aligned}$$

from which (*i.e.*, when possessed of all moments) one can recover $p(x)$ by inverse Fourier transformation:

$$p(x) = \tfrac{1}{2\pi} \int e^{-ixt} \varphi(t)dt$$

While the moments $\{m_1, m_2, \ldots\}$ provide a characteriztion of $p(x)$, so also do the "cumulants" $\{\varkappa_1, \varkappa_2, \ldots\}$, which were introduced in 1903 by the Danish astronomer and statistician T. N. Thiele (1838–1910), and arise from writing

[20] Prior to 1969 the Reed College physics curriculum provided no systematic account of the probability and statistical theory most relevant to physicists, and no thermodynamics beyond that presented in the introductory course. To remedy this defect, I introduced a "Statistical Physics & Thermodynamics" course (which evolved into the present "Thermal Physics" course, required of all majors). I borrow here from Chapter I of the notes (1969–1972) written for that initial series of lectures.

[21] This is certainly a strong assumption, since the Cauchy distribution

$$p(x) = \frac{1}{\pi(1+x^2)}$$

—first studied by Poisson (1824) and only later by Cauchy (1853), known among physicists as the Lorentz or Breit-Wigner distribution—possesses moments of *no* integral order, even though when plotted it looks very "normal."

$$\varphi(t) = \sum_{k=0}^{\infty} \tfrac{1}{k!} m_k (it)^k = e^{\psi(t)} \tag{24.1}$$

$$\psi(t) = \sum_{k=1}^{\infty} \tfrac{1}{k!} \varkappa_k (it)^k \tag{24.2}$$

where the lower limit of the latter sum is (not 0 but) 1 because $\varphi(0) = m_0 = 1$ implies $\psi(0) = \varkappa_0 = 0 \pmod{2\pi}$. When with *Mathematica*'s assistance we expand $e^{\psi}$ we obtain

$$\left.\begin{aligned} m_0 &= 1 \\ m_1 &= \varkappa_1 \\ m_2 &= \varkappa_1^2 + \varkappa_2 \\ m_3 &= \varkappa_1^3 + 3\varkappa_1\varkappa_2 + \varkappa_3 \\ m_4 &= \varkappa_1^4 + 6\varkappa_1^2\varkappa_2 + 4\varkappa_1\varkappa_3 + 3\varkappa_2^2 + \varkappa_4 \\ &\vdots \end{aligned}\right\} \tag{25}$$

which describe moments of ascending order in terms of cumulants of ascending order.[22]

It is in connection with the systematic derivation of (25) that Bell enters the picture.[23] Equations (24), which concern the description of moments $\{m_n\}$ in terms of cumulants $\{\varkappa_k\}$, present an instance of the general problem of expanding a composite function

$$F(x) = \Phi(f(x)) = \sum_{n=0}^{\infty} \tfrac{1}{n!} F_n x^n$$

the solution of which hinges on one's ability to construct the $n^{\text{th}}$ derivative of such a function. As it happens, in 1958, when I confronted another instance of that problem and had gone to the library in quest of a related paper,[24] I looked

[22] These equations assume a simpler appearance when expressed in terms of "centered moments" (or "moments about the mean")

$$\mu_n = \langle (x-m)^n \rangle = \sum_{k=0}^{n} \binom{n}{k} m_k (-m)^{n-k} \quad : \quad m = m_1$$

See the "Statistical Physics" notes[20], page 37.

[23] Eric Temple Bell (1883–1960) wrote, among other things, *The Development of Mathematics* (1940), *Men of Mathematics* (1937) and much science fiction. He was a guest of Robert Rosenbaum and the Reed College mathematics faculty in the spring of 1953, when I had an opportunity to meet him. My autographed copy of *Men of Mathematics* was important to me during my early career. But I loaned it to a student and—alas!—never saw it again.

[24] A. Dresden, "The derivatives of composite functions," Amer. Math. Monthly **50**, 9 (1943).

into the current issue of that journal and encountered an item that answered precisely to my needs: V. F. Ivanoff[25] challenged readers to show that the $n^{\text{th}}$ derivative $F^{(n)}(x)$ of the composite function $F(x) = \Phi(f(x))$ can be described

$$\begin{vmatrix} f'D & f''D & f'''D & f''''D & \cdots & f^{(n)}D \\ -1 & f'D & 2f''D & 3f'''D & \cdots & \binom{n-1}{1}f^{(n-1)}D \\ 0 & -1 & f'D & 3f''D & \cdots & \binom{n-1}{2}f^{(n-2)}D \\ 0 & 0 & -1 & f'D & \cdots & \binom{n-1}{3}f^{(n-3)}D \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & f'D \end{vmatrix} \Phi(f) \tag{26}$$

where $f^{(k)} \equiv (\frac{d}{dx})^k f(x)$ and $D^k\Phi(f) \equiv (\frac{d}{df})^k\Phi(f)$. I was able to develop several alternative demonstrations and associated recursion relations, and over the years have drawn upon (26) and its corollaries to crack a wide assortment of problems.[26] Only much later did I discover that (26) has come to be known as "Faà di Bruno's formula," though (if not yet in determinantal form) it first appeared (1800) in a calculus treatise by Louis François Antoine Arbogast (1759–1803). Faà di Bruno[27] (1825 –1888) entered the picture with the publication in 1855 and 1857 of a pair of 2-page papers in which the determinant (26) first appeared, and a text (1876) that finally provided detailed arguments.

Of exceptional simplicity—and importance, because of its very frequent occurance—is the case $\Phi(f) = e^f$. Then $D\Phi = \Phi$ and the $D$ operators can be dropped from (26). Equations (24) provide an instance of just such a case, so by (26) we have (recall $\varkappa_0 = 0$)

---

[25] Advanced Problem No. 4782, Amer. Math. Monthly **65**, 212 (1958).

[26] See "Foundations and applications of the Schwinger action principle," Appendix A, pages 128–157 (Brandeis University dissertation, 1960); "Some applications of an elegant formula due to V. F. Ivanoff," notes for a seminar presented 28 May 1969 to the Applied Math Club at Portland State University, reprinted in my *Collected Seminars 1963-1970*; "Algorithm for the efficient evaluation of the trace of the inverse of a matrix," (December 1996).

[27] Francesco Faà di Bruno began his career as a military officer, but resigned his commission to study mathematics in Paris, where his dissertation was directed by Cauchy and he became a close friend of Hermite. He taught at the University of Turin (Peano was one of his students), but at the advanced age of 51 was ordained as a priest, and devoted himself to finding ways to relieve the plight of maids, domestic servants, unmarried mothers and prostitutes; he founded a publish house that employed orphaned girls as mathematical typesetters. In 1988 he was beatified by Pope John Paul II. For a detailed account of Faà di Bruno's work (and the work of others) as it relates to his eponymous formula, see Warren Johnson, "The curious history of Faà di Bruno's formula," Amer. Math. Monthly **109**, 217–234 (2002), which is available as a free pdf download on the web.

$$\begin{aligned}
m_0 &= 1\\
m_1 &= \varkappa_1\\
m_2 &= \begin{vmatrix} \varkappa_1 & \varkappa_2\\ -1 & \varkappa_1 \end{vmatrix}\\
m_n &= \begin{vmatrix} \varkappa_1 & \varkappa_2 & \varkappa_3 & \varkappa_4 & \cdots & \varkappa_n\\ -1 & \varkappa_1 & 2\varkappa_2 & 3\varkappa_3 & \cdots & \binom{n-1}{1}\varkappa_{n-1}\\ 0 & -1 & \varkappa_1 & 3\varkappa_2 & \cdots & \binom{n-1}{2}\varkappa_{n-2}\\ 0 & 0 & -1 & \varkappa_1 & \cdots & \binom{n-1}{3}\varkappa_{n-3}\\ \vdots & \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & 0 & \cdots & \varkappa_1 \end{vmatrix}
\end{aligned}$$

which give back (25).

To invert (25) (*i.e.*, to describe cumulants in terms of moments) we work from the functional inverse of (24):

$$\sum_{n=0}^{\infty} \tfrac{1}{n!}\varkappa_n\lambda^n = \log\Big[\sum_{k=0}^{\infty}\tfrac{1}{k!}m_k\lambda^k\Big]$$

By (26) we have

$$\begin{aligned}
\varkappa_0 &= \log m_0 = \log 1 = 0\\
\varkappa_n &= \begin{vmatrix} m_1D & m_2D & m_3D & m_4D & \cdots & m_nD\\ -1 & m_1D & 2m_2D & 3m_3D & \cdots & \binom{n-1}{1}m_{n-1}D\\ 0 & -1 & m_1D & 3m_2D & \cdots & \binom{n-1}{2}m_{n-2}D\\ 0 & 0 & -1 & m_1D & \cdots & \binom{n-1}{3}m_{n-3}D\\ \vdots & \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & 0 & \cdots & m_1D \end{vmatrix} \varPhi(f)\Big|_{\lambda=0}
\end{aligned}$$

with $\varPhi(f) = \log f$ but now—as in all non-exponential cases—cannot abandon but must take into explicit account the action of the $D$-operators. Expanding the determinants, we have

$$\begin{aligned}
\varkappa_1 &= \big[m_1D\big]\log f\big|_{\lambda=0}\\
\varkappa_1 &= \big[m_2D + m_1^2D^2\big]\log f\big|_{\lambda=0}\\
\varkappa_2 &= \big[m_3D + 3m_1m_2D^2 + m_1^3D^3\big]\log f\big|_{\lambda=0}\\
\varkappa_3 &= \big[m_4D + (3m_2^2 + 4m_1m_3)D^2 + 6m_1^2m_3D^3 + m_1^4D^4\big]\log f\big|_{\lambda=0}
\end{aligned}$$

Using $D^k\log(f) = (-)^{k-1}(k-1)!f^{-k}$ which becomes $(-)^{k-1}(k-1)!$ at $\lambda = 0$, we find

$$\left.\begin{aligned}
\varkappa_1 &= m_1\\
\varkappa_2 &= m_2 - m_1^2\\
\varkappa_3 &= m_3 - 3m_1m_2 + 2m_1^3\\
\varkappa_4 &= m_4 - 4m_1m_3 + 12m_1^2m_2 - 3m_2^2 - 6m_1^4
\end{aligned}\right\}\tag{27}$$

which agree precisely with the results produced by the command

$$\texttt{Series[Log[}1+m_1\lambda+\tfrac{1}{2}m_2\lambda^2+\tfrac{1}{6}m_3\lambda^3+\tfrac{1}{24}m_4\lambda^4\texttt{],}\{\lambda,0,4\}\texttt{]}$$

We note that terms on the right side of (25)—ditto those on the right side of (27)—associate in an obvious way with the partitions of the subscripted index:

$$\begin{aligned}
\texttt{Partitions[1]} &= \{1\}\\
\texttt{Partitions[2]} &= \{2\},\{1,1\}\\
\texttt{Partitions[3]} &= \{3\},\{2,1\},\{1,1,1\}\\
\texttt{Partitions[4]} &= \{4\},\{3,1\},\{2,2\},\{2,1,1\},\{1,1,1,1\}
\end{aligned}$$

The following remarks refer to the elegant iceberg of which that observation exposes only the tip. We look to this slight notational variant of (24):

$$\hat{G}(t,u) = \exp\Big\{u\sum_{j=1}^{\infty}x_j t^j\Big\} \tag{28.1}$$

$$= 1+\sum_{n=1}^{\infty}t^n\Big\{\sum_{k=1}^{n}\tfrac{1}{k!}u^k\hat{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})\Big\} \tag{28.2}$$

where the $\hat{B}_{n,k}$ are "ordinary partial Bell polynomials"[28] and $\hat{G}(t,u)$ is their generating function. If we use

$$\texttt{G=Series[Exp[u}\textstyle\sum_{j=1}^{10}x_j t^j\texttt{],\{t,0,8\},\{u,0,8\}]//Simplify}$$

to construct leading terms in the double series, and use (say)

```
6!SeriesCoefficient[G,{6}]
```

to pull out the coefficient of $t^6$ we obtain

$$\begin{aligned}
\sum_{k=1}^{6}\tfrac{1}{k!}u^k\hat{B}_{6,k}&(x_1,x_2,\dots,x_{6-k+1})\\
&=\big\{ux_6+\tfrac{1}{2!}u^2(x_3^2+2x_2x_4+2x_1x_5)\\
&\qquad+\tfrac{1}{3!}u^3(x_2^3+6x_1x_2x_3+3x_1^2x_4)\\
&\qquad+\tfrac{1}{4!}u^4(6x_1^2x_2^2+4x_1^3x_3)+\tfrac{1}{5!}u^5(5x_1^4x_2)+\tfrac{1}{6!}u^6(x_1^6)\,\big\}
\end{aligned}$$

The deployment of subscripts is seen to conform precisely to $p(6,k)$, the partitions of 6 into $k$ parts (produced by `IntegerPartitions[6,{k}]`)

$$\begin{aligned}
p(6,1) &\ :\ \{6\}\\
p(6,2) &\ :\ \{5,1\},\{4,2\},\{3,3\}\\
p(6,3) &\ :\ \{4,1,1\},\{3,2,1\},\{2,2,2\}\\
p(6,4) &\ :\ \{3,1,1,1\},\{2,2,1,1\}\\
p(6,5) &\ :\ \{2,1,1,1,1\}\\
p(6,6) &\ :\ \{1,1,1,1,1,1\}
\end{aligned}$$

[28] Terminology taken from the Wikipedia article "Bell polynomials," where such polynomials are denoted $\hat{B}_{n,k}$ to distinguish them from the "exponential partial Bell polynomials" $B_{n,k}$: see below.

while the fractional coefficients are seen to have the form

$$\frac{k!}{\text{product of factorials of the exponents}}$$

We see in particular that $\hat{B}_{6,3}(x_1, x_2, x_3, x_4)$ can be described

$$\hat{B}_{6,3} = x_2^3 + 6x_1x_2x_3 + 3x_1^2x_4) = 3!\sum_\mu{}'' \frac{x_1^{\mu_1}x_2^{\mu_2}x_3^{\mu_3}x_4^{\mu_4}}{\mu_1!\mu_2!\mu_3!\mu_4!}$$

where the $\sum''$ ranges over all sets $\{\mu_1, \mu_2, \mu_3, \mu_4\}$ such that

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = 3$$
$$\mu_1 + 2\mu_2 + 3\mu_3 + 4\mu_4 = 6$$

*i.e.*, over $\{0, 3, 0, 0\}$, $\{1, 1, 1, 0\}$ and $\{2, 0, 0, 1\}$. Generally

$$\hat{B}_{n,k}(x_1, x_2, \ldots, x_\nu) = \sum_\mu{}'' \frac{k!}{\mu_1!\mu_2!\cdots\mu_\nu!} x_1^{\mu_1}x_2^{\mu_2}\cdots x_\nu^{\mu_\nu} \tag{29}$$
$$\mu_1 + \mu_2 + \cdots + \mu_\nu = k$$
$$\mu_1 + 2\mu_2 + \cdots + \nu\mu_\nu = n$$

where $\nu \equiv n - k + 1$ and $k = 0, 1, 2, \ldots, n$ $(n = 0, 1, 2, \ldots)$. We note that $\hat{B}_{0,0} = 1$ and that $\hat{B}_{n,k}(x_1, x_2, \ldots, x_\nu)$ is homogeneous of degree $k$.

The associated "exponential partial Bell polynomials" $B_{n,k}(x_1, x_2, \ldots, x_\nu)$ are generated by

$$G(t, u) = \exp\Big\{u\sum_{j=1}^\infty \tfrac{1}{j!}x_jt^j\Big\} \tag{30.1}$$

$$= 1 + \sum_{n=1}^\infty \tfrac{1}{n!}t^n\Big\{\sum_{k=1}^n u^kB_{n,k}(x_1, x_2, \ldots, x_{n-k+1})\Big\} \tag{30.2}$$

can be obtained from

$$B_{n,k}(x_1, x_2, \ldots, x_\nu) = \tfrac{n!}{k!}\hat{B}_{n,k}(\tfrac{1}{1!}x_1, \tfrac{1}{2!}x_2, \ldots, \tfrac{1}{\nu!}x_\nu)$$

which gives, for example,

$$\begin{aligned} B_{6,3}(x_1, x_2, x_3, x_4) &= 120\big((\tfrac{1}{2}x_2)^3 + 6x_1(\tfrac{1}{2}x_2)(\tfrac{1}{6}x_3) + 3x_1^2(\tfrac{1}{24}x_4)\big) \\ &= 15x_2^3 + 60x_1x_2x_3 + 15x_1^2x_4 \end{aligned}$$

The coefficients in such expressions can be obtained from theory having to do with the "partitions of sets."[28] In the present instance

$$15 = \begin{cases}\text{number of ways to partition a 6-element set}\\ \text{into subsets of sizes 2, 2 and 2}\end{cases}$$

$$60 = \begin{cases}\text{number of ways to partition a 6-element set}\\ \text{into subsets of sizes 1, 2 and 3}\end{cases}$$

$$15 = \begin{cases}\text{number of ways to partition a 6-element set}\\ \text{into subsets of sizes 1, 1 and 4}\end{cases}$$

I indicate how *Mathematica* can be coaxed to producd those numbers. Let the elements of the 6-element set be $\{1,2,3,4,5,6\}$. The command

```
W=KSetPartitions[{1,2,3,4,5,6},3]
```

produces a list of 90 such partitions, of which the first is $\{\{1\},\{2\},\{3,4,5,6\}\}$ and the last is $\{\{1,4\},\{2,5\},\{3,6\}\}$. The command

```
X=Table[Table[Length[Part[W[[j]],k]],{k,1,3}],{j,1,Length[W]]
```

tabulates the lengths of the respective parts of the partitions listed in `W`. The command

```
Y=Table[Sort[Part[X,k]],{k,1,Length[X]]
```

places the members of each 3-member set in that 90-set list in natural order, and the command `Tally[Y]` announces

$$\{\{2,2,2\},15\},\quad \{\{1,2,3\},60\},\quad \{\{1,1,4\},15\}$$

—as claimed.

If in (30) we set $u=1$ we obtain the generator of the "*complete* exponential Bell polynomials"

$$\begin{aligned} G(t,u) &= \exp\Big\{\sum_{j=1}^{\infty}\tfrac{1}{j!}x_j t^j\Big\} \\ &= 1+\sum_{n=1}^{\infty}\tfrac{1}{n!}t^n\Big\{\sum_{k=1}^{n}B_{n,k}(x_1,x_2,\ldots,x_{n-k+1})\Big\} \\ &= \sum_{n=0}^{\infty}\tfrac{1}{n!}t^n B_n(x_1,x_2,\ldots,x_n) \end{aligned}$$

of which the first few (as constructed by *Mathematica*) are

$$\begin{aligned} B_0 &= 1 \\ B_1(x_1) &= x_1 \\ B_2(x_1,x_2) &= x_1^2+x_2 \\ B_3(x_1,x_2,x_3) &= x_1^3+3x_1x_2+x_3 \\ B_4(x_1,x_2,x_3,x_4) &= x_1^4+6x_1^2x_2+3x_2^2+4x_1x_3+x_4 \\ B_5(x_1,x_2,x_3,x_4,x_5) &= x_1^5+10x_3x_3+15x_1x_2^2+10x_1^2x_3+10x_2x_3+5x_1x_4+x_5 \end{aligned}$$

They can be described determinantally and possess a rich variety of algebraic interrelationships which I will not linger to describe. When all $x_i$ are set equal to $x$ (*i.e.*, when all subscripts are abandoned) they become "Bell polynomials"

$$\begin{aligned} B_0(x) &= 1 \\ B_1(x) &= x \\ B_2(x) &= x+3x^2+x^3 \\ B_4(x) &= x+7x^2+6x^3+x^4 \\ B_5(x) &= x+15x^2+25x^3+10x^4+x^5 \end{aligned}$$

which can be generated by developing $\exp[(e^t-1)x]$ in powers of $t$

$$\exp[(e^t-1)x] = \sum_{n=0}^{\infty} \tfrac{1}{n!} B_n(x)\, t^n \tag{31}$$

and which at $x = 1$ produce "Bell numbers" $B_n$.[29] Bell polynomials $B_n(x)$ inherit properties from the complete exponential Bell polynomials. One has, for example,

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) B_{n-k}(y)$$

which is a corollary of

$$\begin{aligned} &B_n(x_1+y_1,\ldots,x_n+y_n)\\ &\qquad = \sum_{k=1}^{n} \binom{n}{k} B_k(x_1,\ldots,x_k) B_{n-k}(y_1,\ldots,y_{n-k}) \end{aligned}$$

and—because exponents have become subscripts—resembles a $q$-analog of

$$(x+y) = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

We note in the latter connection that while the $\sum$ in

$$(x_1+x_2+\cdots+x_\nu)^k = \sum_{\mu}{}' \binom{k}{\mu_1,\mu_2,\ldots,\mu_\nu} x_1^{\mu_1} x_2^{\mu_2}\cdots x_\nu^{\mu_\nu}$$

is subject to the single constraint $\mu_1+\mu_2+\cdots+\mu_n = k$ (*i.e.*, it ranges over all permutations of all partitions of $k$), the $\sum$ in (29)—which can be written

$$\hat{B}_{n,k}(x_1,x_2,\ldots,x_\nu) = \sum_{\mu}{}'' \binom{k}{\mu_1,\mu_2,\ldots,\mu_\nu} x_1^{\mu_1} x_2^{\mu_2}\cdots x_\nu^{\mu_\nu}$$

—is subject also to a second constraint $\mu_1+2\mu_2+\cdots+\nu\mu_\nu = n$, so can be expected to involve generally many fewer terms.

The Bell numbers—which were seen at (31) to be generated by $\exp[(e^t-1)]$ —are, like all aspects of this subject, replete with combinatorial/permutational/partional connections.[30] Of which I provide only a single example: the command

[29] In my Version 7 of *Mathematica* the commands `Bell[n,x]` and `Bell[n]` produce $B_n(x)$ and $B_n$. Subsequent versions (Version 8, released in 2010, or later) provide a command `BellY[n,k,`$\{x_1,x_2,\ldots,x_{n-k+1}\}$`]` that produces incomplete Bell polynomials of some flavor: a web site provides the example `BellY[4,2,`$\{x_1,x_2,x_3\}$`]` $= 3x_2^2+4x_1x_3$.

[30] For a good survey, see the Wikipedia article "Bell numbers."

```
Flatten[Table[KSetPartitions[{1,2,3},k],{k,1,3}],1]
```

exhibits all the ways of partitioning the set $\{1,2,3\}$ into subsets:

$$\{1,2,3\}$$
$$\{\{1\},\{2,3\}\},\quad \{\{2\},\{1,3\}\},\quad \{\{3\},\{1,2\}\}$$
$$\{\{1\},\{2\},\{3\}\}$$

Of which there are $B_3 = 5$. More generally, the command

```
Length[Flatten[Table[KSetPartitions[{1,2,3,..,n},k],{k,1,n}],1]]
```

—which constructs and then counts all the distinct partitions of $\{1,2,3,\ldots,n\}$ —always responds `True` when set equal (`==`) to `BellB[n]`, though it sometimes takes a while; the Bell numbers become very large very fast:

$$\begin{aligned} B_5 &= 52\\ B_{10} &= 115975\\ B_{15} &= 1382958545\\ B_{20} &= 51724158235372 \end{aligned}$$

**Faà di Bruno, revisited.** From (26) we have (in the illustrative case $n = 5$)

$$\tfrac{d^5}{dx^5}\,\Phi(f(x)) = \begin{vmatrix} f_1D & f_2D & f_3D & f_4D & f_5D\\ -1 & f_1D & 2f_2D & 3f_3D & 4f_4D\\ 0 & -1 & f_1D & 3f_2D & 6f_3D\\ 0 & 0 & -1 & f_1D & 4f_2D\\ 0 & 0 & 0 & -1 & f_1D \end{vmatrix}\Phi(f(x))$$

Expanding the determinant (as in the case $n = 4$ we did already on page 16), we have

$$\Big\{f_5D + (10f_2f_3 + 5f_1f_4)D^2 + (15f_1f_2^2 + 10f_1^2f_3)D^3 \\ + 10f_1^3f_2D^4 + f_1^5D^5\Big\}\,\Phi(f(x))$$

Working now from (30), we look in particular to the coefficient of $\frac{1}{5!}t^5$ and find

$$\begin{aligned} &\sum_{k=1}^{5} B_{5,k}(x_1,x_2,\ldots,x_{5-k+1})u^k\\ &\quad= x_5u + (10x_2x_3 + 5x_1x_4)u^2 + (15x_1x_2^2 + 10x_1^2x_3)u^3 + 10x_1^3x_2u^4 + x_1^5u^5\\ &\quad= B_{5,1}(x_1,x_2,x_3,x_4,x_5)u^1 + B_{5,2}(x_1,x_2,x_3,x_4)u^2\\ &\qquad\qquad + B_{5,3}(x_1,x_2,x_3)u^3 + B_{5,4}(x_1,x_2)u^4 + B_{5,5}(x_1)u^5 \end{aligned}$$

Compare the $D$-series with the $u$-series and it becomes evident that we have in the case $n = 5$ (and can expect to have in general)

$$\big(\tfrac{d}{dx}\big)^n\Phi(f(x)) = \sum_{k=1}^{n} B_{n,k}(f_1,\ldots,f_\nu)\cdot D^k\Phi(f) \tag{32.1}$$

$$= \sum_{k=1}^{n}\Big\{\sum_{\mu}{}''\binom{n}{\mu_1,\mu_2,\ldots,\mu_\nu}f_1^{\mu_1}f_2^{\mu_2}\cdots f_\nu^{\mu_\nu}\Big\}\big(\tfrac{d}{df}\big)^k\Phi(f) \tag{32.2}$$

where again: $\nu = n - k + 1$, $\sum_{s=1}^{\nu} \mu_s = k$, $\sum_{s=1}^{\nu} s\mu_s = n$. Equation (32.1) provides an elegantly succinct and efficient formulation of "Faà di Bruno's formula," seen now to be equivalent to "Faà di Bruno's determinantal formula" (26). I conclude with discussion of an uncommon application[26] of the latter that I have, over the years, found to be particularly useful.

**Application to matrix theory.** Let $\mathbb{A} = \mathbb{I} - x\mathbb{M}$ be a non-singular square matrix. From the generalized spectral decomposition of $\mathbb{A}$ one can construct[31] a matrix $\mathbb{B} = \log \mathbb{A}$ such that $\mathbb{A} = e^{\mathbb{B}}$ and establish that $\det \mathbb{A} = e^{\operatorname{tr} \mathbb{B}}$. Drawing formally upon the series

$$\log(1-z) = -z - \tfrac{1}{2}z^2 - \tfrac{1}{3}z^3 - \tfrac{1}{4}z^4 - \cdots \quad : \quad z^2 < 1$$

we expect to have

$$F(x) = \sum_{n=0}^{\infty} \tfrac{1}{n!} F_n x^n \equiv \det(\mathbb{I} - x\mathbb{M}) = e^{f(x)}$$

with

$$\begin{aligned} f(x) &= -T_1 x - \tfrac{1}{2}T_2 x^2 - \tfrac{1}{3}T_3 x^3 - \tfrac{1}{4}T_4 x^4 - \cdots \quad : \quad T_n = \operatorname{tr} \mathbb{M}^n \\ &= f_0 + \tfrac{1}{1!}f_1 x^1 + \tfrac{1}{2!}f_2 x^2 + \tfrac{1}{3!}f_3 x^3 + \tfrac{1}{4!}f_4 x^4 + \cdots \end{aligned}$$

$$f_0 = 0 \quad \text{and} \quad f_k = -(k-1)!\,T_k \;:\; k = 1, 2, 3, \ldots$$

From (26)—noting that $e^{f(0)} = 1$—we obtain

$$\begin{aligned} F_n &= \begin{vmatrix} f_1 & f_2 & f_3 & f_4 & \cdots & f_n \\ -1 & f_1 & 2f_2 & 3f_3 & \cdots & \binom{n-1}{1} f_{n-1} \\ 0 & -1 & f_1 & 3f_2 & \cdots & \binom{n-1}{2} f_{n-2} \\ 0 & 0 & -1 & f_1 & \cdots & \binom{n-1}{3} f_{n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & f_1 \end{vmatrix} \\ &= (-)^n \begin{vmatrix} T_1 & T_2 & 2T_3 & 6T_4 & \cdots & (n-1)!T_n \\ 1 & T_1 & 2T_2 & 6T_2 & \cdots & \binom{n-1}{1}(n-2)!T_{n-1} \\ 0 & 1 & T_1 & 3T_2 & \cdots & \binom{n-1}{2}(n-3)!T_{n-2} \\ 0 & 0 & 1 & T_1 & \cdots & \binom{n-1}{3}(n-4)!T_{n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & T_1 \end{vmatrix} \end{aligned}$$

Determinants of the latter design can be brought to the more attractive form

[31] See "Working notes concerning Lester Lipsky's *A Markov Model of Maxwell Boltzmann & Bose Einstein Statistics*," (November 2016), page 6.

$$F_0 = 1, \quad F_n = (-)^n \begin{vmatrix} T_1 & T_2 & T_3 & T_4 & \cdots & & T_n \\ 1 & T_1 & T_2 & T_3 & \cdots & & T_{n-1} \\ 0 & 2 & T_1 & T_2 & \cdots & & T_{n-2} \\ 0 & 0 & 3 & T_1 & \cdots & & T_{n-3} \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n-1 & T_1 \end{vmatrix} \tag{33}$$

by elementary manipulations, the pattern of which is illustrated below.[32] From (33) we obtain

$$\begin{aligned}
&F_0 = 1\\
&F_1 = -\{T_1\}\\
&F_2 = +\{T_1^2 - T_2\}\\
&F_3 = -\{T_1^3 - 3T_1T_2 + 2T_3\}\\
&F_4 = +\{T_1^4 - 6T_1^2T_2 + 3T_2^2 + 8T_1T_3 - 6T_4\}\\
&F_5 = -\{T_1^5 - 10T_1^3T_2 + 15T_1T_2^2 + 20T_1^2T_3 - 20T_2T_3 - 30T_1T_4 + 24T_5\}\\
&\quad\vdots
\end{aligned}$$

all of which can be formulated in terms of incomplete Bell polynomials, and which—though obtained here with the assistance of *Mathematica*—can be obtained from the simple recursion relations that I have described elsewhere.[26]

Of course, if $\mathbb{M}$ is $m\times m$ then $F(x) = \det(\mathbb{I} - x\mathbb{M})$ is a polynomial of degree $m$, which entails $F_{n>m} = 0$. I discuss now how this comes about. The characteristic polynomial of $\mathbb{M}$ can be written

$$\begin{aligned}
\det(\mathbb{M} - \lambda\mathbb{I}) &= (-)^m\lambda^m \det(\mathbb{I} - \lambda^{-1}\mathbb{M})\\
&= (-\lambda)^m \sum_{n=0}^{\infty} \tfrac{1}{n!} F_n \lambda^{-n}\\
&= \sum_{n=0}^{\infty} \tfrac{1}{n!} \Delta_n (-\lambda)^{m-n} \quad \text{with} \quad \Delta_n = (-)^n F_n
\end{aligned} \tag{34}$$

Look, for example, to the case $m = 3$ :

$$\begin{aligned}
\det(\mathbb{M} - \lambda\mathbb{I}) &= \Delta_0(-\lambda)^3 + \Delta_1(-\lambda)^2 + \tfrac{1}{2!}\Delta_2(-\lambda)^1 + \tfrac{1}{3!}\Delta_3(-\lambda)^0\\
&\qquad\qquad + \sum_{n=4}^{\infty} \tfrac{1}{n!}\Delta_n(-\lambda)^{3-n}\\
&= -\lambda^3 + T_1\lambda^2 - \tfrac{1}{2}\{T_1^2 - T_2\}\lambda + \tfrac{1}{6}\{T_1^3 - 3T_1T_2 + 2T_3\}\\
&\qquad - \tfrac{1}{24}\{T_1^4 - 6T_1^2T_2 + 3T_2^2 + 8T_1T_3 - 6T_4\}\lambda^{-1} + \cdots
\end{aligned}$$

---

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$$\begin{vmatrix} T_1 & T_2 & 2T_3 \\ 1 & T_1 & 2T_2 \\ 0 & 1 & T_1 \end{vmatrix} = \tfrac{1}{2}\begin{vmatrix} T_1 & T_2 & 2T_3 \\ 1 & T_1 & 2T_2 \\ 0 & 2 & 2T_1 \end{vmatrix} = \begin{vmatrix} T_1 & T_2 & T_3 \\ 1 & T_1 & T_2 \\ 0 & 2 & T_1 \end{vmatrix}, \textit{ etc.}$$

which at $\lambda = 0$ would give $\det \mathbb{M} = \frac{1}{6}\{T_1^3 - 3T_1T_2 + 2T_3\} + \infty$ were it not the case that

$$\Delta_p = 0 \quad : \quad p > \text{dimension } m$$

which is a direct but non-obvious implication of the fact that $\varphi(\lambda) \equiv \det(\mathbb{M} - \lambda\mathbb{I})$ is a polynomial of degree $m$, and—as shown below—follows from the Cayley-Hamilton theorem:

$$\varphi(\mathbb{M}) = \sum_{n=0}^{m} (-)^{(m-n)} \tfrac{1}{n!}\Delta_n \mathbb{M}^{m-n} = \mathbb{O} \tag{35.1}$$

From (35.1) we obtain

$$\text{tr}[\varphi(\mathbb{M})\,\mathbb{M}^p] = \sum_{n=0}^{m} (-)^{(m-n)} \tfrac{1}{n!}\Delta_n T_{m-n+p} = 0 \quad : \quad p = 0, 1, 2, \ldots \tag{35.2}$$

But the $\Delta_n$ satisfy the recursion relation[33]

$$\Delta_n = \sum_{j=0}^{n} (-)^{j+1} \frac{(n-1)!}{(n-j)!} T_j \Delta_{n-j} \tag{36}$$

so

$$\Delta_{m+p} = (m+p-1)! \sum_{j=0}^{m+p} (-)^{j+1} \frac{1}{(m+p-j)!} T_j \Delta_{m+p-j}$$

which (set $m + p - j = n$, $j = m - n + p$) can be written

$$\begin{aligned} \Delta_{m+p} &= (m+p-1)!(-)^{p+1} \underbrace{\sum_{n=m+p}^{0} (-)^{m-n} \tfrac{1}{n!}\Delta_n T_{m-n+p}}_{0 \text{ by Cayley-Hamilton (35.2)}} \\ &= 0 \text{ when } m+p \text{ exceeds the dimension } m \text{ of } \mathbb{M} \end{aligned}$$

Which checks out: *Mathematica* confirms that if

$$\mathbb{M} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

then

$$\begin{aligned} \Delta_4 &= \{T_1^4 - 6T_1^2T_2 + 3T_2^2 + 8T_1T_3 - 6T_4\} = 0 \\ \Delta_5 &= \{T_1^5 - 10T_1^3T_2 + 15T_1T_2^2 + 20T_1^2T_3 - 20T_2T_3 - 30T_1T_4 + 24T_5\} = 0 \\ &\vdots \end{aligned}$$

[33] The proof[26] proceeds inductively from (33).

The resuls developed above—relatively unfamilar though they are—often prove quite useful. For example, we have in (34) a trace-wise construction of the characteristic polynomial of $\mathbb{M}$, and in

$$\det \mathbb{M} = \tfrac{1}{m!}\Delta_m(T_1, T_2, \ldots, T_m)$$

a trace-wise construction of the determinant of the $m \times m$ matrix $\mathbb{M}$.

If $\mathbb{M}$ is antisymmetric (call it $\mathbb{A}$ to underscore the point) then great simplifications result from the fact that $T_{\text{odd}} = 0$, and from (33) we have

$$\Delta_n = \begin{vmatrix} 0 & T_2 & 0 & T_4 & \cdots & & T_n \\ 1 & 0 & T_2 & 0 & \cdots & & T_{n-1} \\ 0 & 2 & 0 & T_2 & \cdots & & T_{n-2} \\ 0 & 0 & 3 & 0 & \cdots & & T_{n-3} \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n-1 & T_1 \end{vmatrix}$$

where only terms of the form $T_{\text{even}}$ appear in the last column. Thus

$$\begin{aligned} \Delta_0 &= 1 \\ \Delta_1 &= 0 \\ \Delta_2 &= -T_2 \\ \Delta_3 &= 0 \\ \Delta_4 &= 3T_2^2 - 6T_4 \\ \Delta_5 &= 0 \\ \Delta_6 &= -15T_2^3 + 90T_2T_4 - 120T_6 \end{aligned}$$

which—though here computed directly—could have been read off from the descriptions of $F_n = (-)^n\Delta_n$ presented on page 23. It is a familiar fact that when the determinant of such matrices $\mathbb{A}$ do not automatically vanish they are automatic perfect squares (of the so-called "Pfaffian"):

$$\det \mathbb{A} = \begin{cases} 0 & : \text{ odd-dimensional cases} \\ [\mathrm{Pf}(\mathbb{A})]^2 & : \text{ even-dimensional cases} \end{cases}$$

Suppose, for example, that $\mathbb{A}$ is 4-dimensional and antisymmetric. Then

$$\begin{aligned} T_2 &= -2(a_{12}^2 + a_{13}^2 + a_{14}^2 + a_{23}^2 + a_{24}^2 + a_{34}^2) \\ T_4 &= (a_{12}^2 + a_{13}^2 + a_{14}^2)^2 + 2(a_{13}a_{23} + a_{14}a_{24}) + 8 \text{ similar terms} \end{aligned}$$

which when introduced into $\det \mathbb{A} = \frac{1}{4!}\Delta_4(T_2, T_4)$ give

$$\det \mathbb{A} = \tfrac{1}{4!}(3T_2^2 - 6T_4) = (\underbrace{a_{14}a_{23} - a_{13}a_{24} + a_{12}a_{34}}_{\text{Pfaffian of } \mathbb{A}})^2$$

Such matrices $\mathbb{A}$ can be construed to be generators of rotations $\mathbb{R} = e^{\mathbb{A}}$ in 4-space. The characteristic polynomial of $\mathbb{A}$ can be written

$$\begin{aligned} \det(\mathbb{A} - \lambda\mathbb{I}) &= \Delta_0\lambda^4 + \tfrac{1}{2!}\Delta_2\lambda^2 + \tfrac{1}{4!}\Delta_4\lambda^0 \\ &= \lambda^4 + 2\alpha\lambda^2 + \beta^2 \end{aligned}$$

where

$$\begin{aligned} 2\alpha &= \tfrac{1}{2!}\Delta_2 = -\tfrac{1}{2}T_2 = (a_{12}^2 + a_{13}^2 + a_{14}^2 + a_{23}^2 + a_{24}^2 + a_{34}^2) > 0 \\ \beta^2 &= \tfrac{1}{4!}\Delta_4 = \tfrac{1}{8}(T_2^2 - 2T_4) \;=\; (a_{14}a_{23} - a_{13}a_{24} + a_{12}a_{34})^2 \geqslant 0 \end{aligned}$$

By the Cayley-Hamilton theorem $\mathbb{A}^4 + 2\alpha\mathbb{A}^2 + \beta^2\,\mathbb{I} = \mathbb{O}$ which can be factored to read

$$[\,\mathbb{A}^2 + (\alpha+\omega)\,\mathbb{I}\,][\,\mathbb{A}^2 + (\alpha-\omega)\,\mathbb{I}\,] = \mathbb{O} \quad \text{with} \quad \omega = \sqrt{\alpha^2 - \beta^2}$$

Assuming for the moment that $\omega \neq 0$, we establish easily that

$$\mathbb{P}_\pm \equiv \pm\tfrac{1}{2\omega}[\,\mathbb{A}^2 + (\alpha \pm \omega)\,\mathbb{I}\,]$$

comprise a complete set of orthogonal projection matrices:

$$\begin{gathered} \mathbb{P}_+ + \mathbb{P}_- = \mathbb{I} \\ \mathbb{P}_+\mathbb{P}_- = \mathbb{O} \\ \mathbb{P}_\pm\mathbb{P}_\pm = \mathbb{P}_\pm \end{gathered}$$

With this apparatus at hand, one is led by straightforward argument[34] to the conclusion that

$$\begin{aligned} \mathbb{R} &= e^{\mathbb{A}} \\ &= (\cos\phi_+ \cdot \mathbb{I} + \sin\phi_+ \cdot \mathbb{Q}_+)\,\mathbb{P}_+ + (\cos\phi_- \cdot \mathbb{I} + \sin\phi_- \cdot \mathbb{Q}_-)\,\mathbb{P}_- \end{aligned} \tag{37}$$

where $\phi_\pm = \sqrt{\alpha \mp \omega}$ and $\mathbb{Q}_\pm = \mathbb{A}/\phi_\pm$. The right side of (37) describes rotations through angles $\phi_\pm$ on a pair of orthogonal planes in 4-space.[35]

If $\mathbb{M}$ is an $m$-dimensional projection matrix (call it $\mathbb{P}$ to underscore the point) that projects onto a $k$-dimensional subspace of $m$-space, then

$$\big\{\mathbb{P}^2 = \mathbb{P} \;\text{ and }\; \mathrm{tr}\,\mathbb{P} = k\big\} \implies T_p = k \;:\; (\text{all } p)$$

and we have

$$\Delta_n = \begin{vmatrix} k & k & k & k & \cdots & & k \\ 1 & k & k & k & \cdots & & k \\ 0 & 2 & k & k & \cdots & & k \\ 0 & 0 & 3 & k & \cdots & & k \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n-1 & k \end{vmatrix} = \frac{k!}{(k-n)!}$$

[34] For details, see page 19 in "Some applications of an elegant formula. . . "[26]

[35] For extension of the argument to the $n$-dimensional case, see pages 14–18 in "Extrapolated interpolation theory" (1997).

giving

$$\begin{aligned}\det(\mathbb{P}-\lambda\mathbb{I}) &= \sum_{n=0}^{m} \tfrac{1}{n!}\Delta_n(-\lambda)^{m-n}\\ &= \sum_{n=0}^{m} \tfrac{1}{n!}\tfrac{k!}{(k-n)!}(-\lambda)^{m-n}\\ &= (-\lambda)^{m-k}\cdot\sum_{n=0}^{m}\tbinom{k}{n}(-\lambda)^{k-n}\\ &= (-\lambda)^{m-k}\cdot(1-\lambda)^k\end{aligned}$$

We conclude (not unexpectedly!) that the eigenvalues of $\mathbb{P}$ are

- 1, with multiplicity $k$, and
- 0, with multiplicity $m-k$.

**Applications to elementary analysis.** One encounters expressions of the form

$$\big[\tfrac{d}{dx}+f'(x)\big]^n g(x)$$

which by the "shift rule" can be written

$$e^{-f(x)}\big(\tfrac{d}{dx}\big)^n e^{f(x)}g(x) = \sum_{k=0}^{n}\tbinom{n}{k}\Big[e^{-f(x)}\big(\tfrac{d}{dx}\big)^k e^{f(x)}\Big]g^{(n-k)}$$

One can bring (26) to the development of the expression [*etc.*]. To avoid notational clutter I look to the case $n=4$:

$$\begin{aligned}\text{RHS} = g^{(4)} + 4f'g^{(3)} + 6\begin{vmatrix} f' & f'' \\ -1 & f'\end{vmatrix} g^{(2)} \qquad\qquad (38.1)\\ + 4\begin{vmatrix} f' & f'' & f''' \\ -1 & f' & 2f'' \\ 0 & -1 & f'\end{vmatrix} g^{(1)} + \begin{vmatrix} f' & f'' & f''' & f'''' \\ -1 & f' & 2f'' & 3f''' \\ 0 & -1 & f' & 3f'' \\ 0 & 0 & -1 & f'\end{vmatrix} g\end{aligned}$$

which in terms of the complete Bell polynomials (page 19) becomes

$$\begin{aligned}= g^{(4)} &+ 4B_1(f')\cdot g^{(3)}\\ &+ 6B_2(f',f'')\cdot g^{(2)}\\ &+ 4B_3(f',f'',f''')\cdot g^{(1)}\\ &+ \ \ B_4(f',f'',f''',f'''')\cdot g \qquad\qquad (38.2)\end{aligned}$$

In the case $g(x)=1$ we recover the Faà di Bruno-Ivanoff determinant $F_4$ that appears on page 22. We can expect interesting results to arise from (38) whenever $f(x)$ has nice derivative properties. For example, in the simplest case ($f(x)=-\frac{1}{2}x^2,\ g(x)=1$) one is led to determinantal/Bell descriptions of the

Hermite polynomials

$$
\begin{aligned}
He_n(x) &= (-)^n e^{\frac{1}{2}x^2}\left(\tfrac{d}{dx}\right)^n e^{-\frac{1}{2}x^2}\\
&= \begin{vmatrix}
x & 1 & 0 & 0 & 0 & \cdots & 0 & 0\\
1 & x & 2 & 0 & 0 & \cdots & 0 & 0\\
0 & 1 & x & 3 & 0 & \cdots & 0 & 0\\
0 & 0 & 1 & x & 4 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \cdots & x & (n-1)\\
0 & 0 & 0 & 0 & 0 & \cdots & 1 & x
\end{vmatrix}\\
&= B_n(x,1,0,0,\ldots,0)\\
&\Downarrow\\
He_1(x) &= x\\
He_2(x) &= x^2-1\\
He_3(x) &= x^3-3x\\
He_4(x) &= x^4-6x^2+3\\
He_5(x) &= x^5-10x^3+15x
\end{aligned}
\tag{39}
$$

The recursion relation

$$He_n(x) = x\,He_{n-1}(x) - (n-1)He_{n-2}(x)$$

and some of the many other properties of the Hermite polynomials can be obtained as quick consequences of (39).

## ADDENDUM

On page 9 I had occasion to remark that

$$\binom{n}{m} = \frac{n(n-1)\cdots(n-m+1)}{1\cdot 2\cdot 3\cdot\cdots\cdot m} \quad : \quad n \text{ and } m \leqslant n \text{ integers}$$

"gives every appearance of being a fraction," though as the answer to a question of the form "In how many ways...?" it must certainly be an integer. That such denominataors are invariably factors of such numerators I conceived to be a number-theoretic proposition which I was unable to demonstrate.[15] Nor am I yet. Ray Mayer has, however, remarked that the fact that binomial coefficients are integers can be proved by induction: if it be granted that the numbers $\binom{n-1}{k}$ on the $(n-1)^{\text{th}}$ row of Pascal's triangle are integers, then it follows from Pascal's identity

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

that so also are the numbers on the $n^{\text{th}}$ row.

This small matter acquires interest from the fact that *Gaussian* binomial coefficients—Gauss's $q$-analogs of ordinary binomial coefficients

$$\begin{aligned}\binom{n}{k}_q &= \frac{[n]_q!}{[k]_q![n-k]_q!}\\ &= \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}\end{aligned}\tag{19.1}$$

—appear on their face to be *ratios* of polynomials. It is certainly not obvious (to me) that in all cases the denominator is a factor of the numerator, though that is, in all specific cases examined, demonstrably the case (according to *Mathematica*). One has, however, this $q$-analog of Pascal's identity

$$\binom{n-1}{k-1}_q + q^k\binom{n-1}{k}_q = \binom{n}{k}_q \tag{22}$$

which is all that is required to construct a $q$-analog of Mayer's proof by induction: if the $(\bullet)_q$ expressions on the left are $q$-polynomials then so also is the $(\bullet)_q$ expression on the right. Pretty... but this line of argument leaves unanswered the question: *Why* is the denominator a factor of the numerator? To which, my instincts tell me, there exists an elegant one-line answer.